Contributions to the Study of Oscillation Properties of the Solutions of Linear Differential Equations of the Second Order.[*]

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Introduction.

The study of boundary problems for linear differential equations of the second order dates back to the time of Euler and D'Alembert, with whom it arose in connection with problems of mathematical physics. Beginning with the fundamental paper of Sturm in 1836, there have been extensive investigations[†] in this field in recent years, notably by Klein, Bôcher, Stekeloff, Kneser, Hilbert and Birkhoff. Since the differential equation of the second order is of such fundamental importance in so many fields, and since similar general problems for equations of higher order can not be handled by processes so far devised, the invention of new methods and further investigation of the nature of solutions find ready justification.

The equation to be studied will be taken in the form

$$\frac{d}{dx}\left(p(x)\,\frac{dy(x)}{dx}\right)+G(x,\,\lambda)\,y(x)\equiv(py_x)_x+G(x,\,\lambda)\,y=0,\qquad 0\leqq x\leqq 1,\tag{1}$$

where $G(x, \lambda)$ is a function depending on a parameter λ. The solution $y(x)$ of this self-adjoint equation shall be subject to the self-adjoint boundary conditions

$$\left.\begin{aligned}&\alpha_1 y(0)+\alpha_2 y_x(0)+\alpha_3 y(1)+\alpha_4 y_x(1)=0,\\&\beta_1 y(0)+\beta_2 y_x(0)+\beta_3 y(1)+\beta_4 y_x(1)=0,\\&p(1)\,(\alpha_1\beta_2-\alpha_2\beta_1)=p(0)\,(\alpha_3\beta_4-\alpha_4\beta_3),\end{aligned}\right\}\tag{2}$$

where the two sets of real coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ are linearly independent. The most important special cases of these boundary conditions are given by

$$y(0)=y(1)=0;\quad y(0)=y_x(1)=0;\quad y_x(0)=y(1)=0;\quad y_x(0)=y_x(1)=0.\tag{2'}$$

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[*] Read before the American Mathematical Society, September 4, 1917.

[†] For existing methods and literature of the subject see Bôcher, *Encyklopädie Mathematischen Wissenschaften*, II A7a, *Proceedings International Congress of Mathematicians*, Vol. I (1912), p. 163, and "Leçons sur les Methodes de Sturm" (1917); Lichtenstein, *Rendiconti del Circolo Matematico di Palermo*, Vol. XXXVIII, p. 113.

There are many interesting questions in regard to this linear problem. Do there exist parameter values λ such that there are solutions of (1) satisfying relations (2)? If so, how many are there, and how are they distributed? What is the nature of the corresponding solutions? How do the solutions vary with change of the coefficients of the equation and of the boundary conditions? When does the totality of solutions form a fundamental set in terms of which functions may be expanded?

Among the methods which have been used in studying the problem are: (1) Differential equations including the use of comparison, approximation and asymptotic expressions; (2) the minimum principle in the calculus of variations; (3) integral equations; (4) the theory of linear algebraic equations in an infinite number of variables;* (5) a limiting process with linear algebraic approximating difference equations (cf. § 1, IV). The methods developed in this memoir would fall under (1) and (2) and may be characterized as a free use of the differentiation of fundamental formulae with regard to the parameters involved in the equations and in the boundary conditions, together with the proof that under the conditions imposed certain integrals are positive.

Exact oscillation theorems for solutions under the boundary conditions (2) have been developed by Birkhoff † for the case of the special equation

$$y_{xx}+G(x,\lambda)y=0,\quad \frac{\partial G}{\partial\lambda}>0,\quad \lim_{\lambda=-\infty} G=-\infty,\quad \lim_{\lambda=+\infty} G=+\infty.$$

Another important special case of equation (1) in which $G(x,\lambda)$ contains the parameter linearly

$$(py_x)_x+(q+\lambda k)y=0, \tag{3}$$

has been studied very extensively. When $k(x)>0$ this equation may be reduced to a form included in that investigated by Birkhoff.

The *definite* case of (3), viz., when one of the integrals

$$\int_0^1 ky^2dx,\quad \int_0^1 (py_x^2-qy^2)\,dx$$

has one sign for all functions $y(x)$ considered, has been discussed in many phases by mathematicians since the time of Sturm. By means of his theory of integral equations Hilbert ‡ established the *existence* of characteristic

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* Lichtenstein, *loc. cit.*

† *Transactions of the American Mathematical Society*, Vol. X (1909), p. 259. It should be remarked that both in the second and third lines from the end of the statement of the principal theorem (p. 269) instead of $p+1$ we should read $p-1$.

‡ "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen" (Teubner, 1912).

parameter values and characteristic solutions of equation (3) for this definite case. As will be shown in § 5 the boundary conditions which he used are normal forms of (2). Oscillation theorems for equation (3), under some of the simple boundary conditions (2′), have been established by various means; among others by setting up the corresponding calculus of variations problem, and interpreting the Jacobi criterion.*

When k has both signs and q is positive in at least a part of the interval and sufficiently large there, it is not necessary that either of the integrals in question be definite. This case, which we shall call the *non-definite*, was first discussed incidentally by the author in a paper † in which he was treating the problem of oscillation theorems for two equations with two parameters. He showed that when the boundary conditions are $y(0)=y(1)=0$, there exists an integer n_1 such that for $n<n_1$ there are *no real solutions* which have n zeros, while for $n>n_1$ there are *at least two*.‡ At that time all the principal results of §§ 2–4 were obtained, but were not published.

For the special equation (3) and $k>0$ the theorems proved by Birkhoff were rediscovered by Haupt § in his dissertation. The oscillation theorem stated in this dissertation for the case that k changes sign is corrected in a later article,|| and by using the methods which I had developed, various oscillation theorems for the general equation (1) are derived. It is also shown by means of expansion theorems that if the non-definite equation (3) be taken in a certain normal form there exists an integer n_2 such that for $n>n_2$ there are precisely two solutions with n zeros and satisfying the boundary conditions (2).

The object of the present memoir is to investigate the conditions to be imposed on $G(x, \lambda)$ regarded as a function of λ, so that definite oscillation theorems for solution of (1) may be determined. In § 2 an attempt is made to bring as close together as possible necessary conditions and sufficient conditions for a limited or unlimited number of oscillations. A criterion for the behavior of the zeros with change of the parameter is obtained in § 3. This permits the development of very general theorems for the unique existence of solutions vanishing at the end points and possessing a prescribed number of zeros, and also for the existence of two, and only two solutions of this nature.

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* *Mathematische Annalen,* Vol. LXVIII (1910), p. 279.

† *Transactions of the American Mathematical Society,* Vol. XIII (1912), p. 22.

‡ That there were *exactly* two when $n>n_1$ was stated by the author in a paper in the *Mathematische Annalen,* Vol. LXXIII, p. 289, in which he was discussing oscillation theorems for three linear equations with three parameters. The subsequent results of the memoir were not affected by this error which was corrected by a note in Vol. LXXIV (1913), p. 312, of the same journal.

§ "Untersuchungen über Oszillationstheoreme" (Teubner, 1911).

|| Haupt, *Mathematische Annalen,* Vol. LXXVI, p. 67.

These theorems contain as special cases all known results in this field, and some new special cases are set forth in detail.

The non-definite case of (3) and (1) is discussed in § 4. The question of whether there may be for a given oscillation number more than two parameter values for which there are solutions of (3) (2) is settled by giving an example in which for any n in an interval n_1, n_2 there are four values of λ corresponding. That there exists an integer n_2 such that for $n > n_2$, there are exactly two solutions is proved by a method entirely different from that of Haupt, and in some particulars it would seem that the resulting theorem is less satisfactory, in others more satisfactory than his. The theory is also extended to cover some corresponding cases of (1). Concerning the complex solutions which correspond to values of $n < n_1$ some theorems are derived.

In the latter half of the memoir a method is developed for obtaining the facts in regard to the solutions of the *general* equation (1) under the general boundary conditions (2). With this end in view § 5 is devoted to a reduction of the boundary conditions to normal forms by means of the usual transformation of the dependent variable y, which leaves the number of zeros unchanged. Each of these three normal forms,

I.	$\sigma y(0)+y_x(0)=0,$	$\tau y(1)+y_x(1)=0;$
II.	$y(0)=hy(1),$	$hp(0)y_x(0)=p(1)y_x(1);$
III.	$y(0)=lp(1)y_x(0),$	$lp(0)y_x(0)=y(1)$

contains one or two parameters σ, τ, h, l. When special values 0 or ∞ are assigned to these parameters, the forms reduce to the simple cases (2′) for which the facts are readily obtainable from the developments of the earlier sections. The first form is of essentially different character from the others. In the latter the parameter h or l is a double-valued function of λ, which in general is real in sub-intervals only, while in the former each of the parameters σ, τ is a single-valued function of the other and of λ, and is real throughout. By letting λ vary, and calculating the rates of change of these parameters and of $G(x, \lambda)$, theorems of oscillation are derived (§§ 6–8) for boundary conditions in each of the normal forms. While detailed results are not given in all cases, these are immediate developments of the fundamental facts ascertained.

Any linear differential equation of the second order

$$\phi y_{xx}+\psi y_x+\theta y=0 \quad \text{or} \quad \phi y_{xx}+\psi y_x+(\lambda\theta_1+\theta_2)y=0, \quad \phi>0,$$

may be thrown into the corresponding self-adjoint form (1) or (3) on multiplying by the function $p(x)=e^{\int_0^x \frac{\psi}{\phi}dx}$; the corresponding self-adjoint boundary

conditions (2) are changed in form only by the substitution $p(0)=1$, $p(1)=e^{\int_0^1 \frac{\psi}{\phi}dx}$.

The oscillation theorems remain unchanged if the variables are subjected to the usual transformations of the dependent and independent variables $y=\eta\bar{y}$, $x=\xi(\bar{x})$ $\left(\eta(x)\neq 0,\ \frac{d\xi}{d\bar{x}}\neq 0\right)$. The resulting equation is of the form

$$\bar{\phi}\bar{y}_{\bar{x}\bar{x}}+\bar{\psi}\bar{y}_{\bar{x}}+\bar{\theta}\bar{y}=0 \quad \text{or} \quad \bar{\phi}\bar{y}_{\bar{x}\bar{x}}+\bar{\psi}\bar{y}_{\bar{x}}+(\lambda\bar{\theta}_1+\bar{\theta}_2)\bar{y}=0,$$

the new boundary conditions for the new interval $\bar{x}_0$, $\bar{x}_1$ being of the form

$$\begin{aligned}
&\bar{\alpha}_1\bar{y}(\bar{x}_0)+\bar{\alpha}_2\bar{y}_{\bar{x}}(\bar{x}_0)+\bar{\alpha}_3\bar{y}(\bar{x}_1)+\bar{\alpha}_4\bar{y}_{\bar{x}}(\bar{x}_1)=0,\\
&\bar{\beta}_1\bar{y}(\bar{x}_0)+\bar{\beta}_2\bar{y}_{\bar{x}}(\bar{x}_0)+\bar{\beta}_3\bar{y}(\bar{x}_1)+\bar{\beta}_4\bar{y}_{\bar{x}}(\bar{x}_1)=0,\\
&e^{\int_{x_1}^{x_2}\frac{\psi}{\phi}d\bar{x}}\,[\bar{\alpha}_1\bar{\beta}_2-\bar{\alpha}_2\bar{\beta}_1]=\bar{\alpha}_3\bar{\beta}_4-\bar{\alpha}_4\bar{\beta}_3,
\end{aligned}$$

as may be shown by computation.* The invariantive property of self-adjointness gives to the results obtained for (1) or (3) a very general character.

§ 1. *Some Properties of Solutions of the Differential Equation.*

In his original memoir Sturm studied the differential equation in the form

$$\frac{d}{dx}\left(K(x,\lambda)\frac{dy}{dx}\right)+\bar{G}(x,\lambda)y=0,$$

where $K>0$ and $\bar{G}$ both depend on a parameter λ. But by a change of variables this may be reduced to the form

$$(py_x)_x+G(x,\lambda)y=0, \tag{4}$$

where $p(x)$ is a positive function independent of λ. We lose nothing in generality by considering this latter equation. Regarded as functions of x the coefficients p, G will be postulated as continuous together with as many derivatives as is desired, while G will be considered as analytic with respect to λ. The usual modifications of the results derived can be written down immediately if less stringent hypotheses are imposed. The trivial solution $y\equiv 0$ will be excluded from the discussion. Some theorems concerning solutions of (4) will now be reviewed.

I. If one zero of a solution of (4) is held fixed, all others are moved nearer to it by a decrease of p or an increase of G. If, for example, $p(x)$ is less than a constant P, and $G(x,\lambda)$ is greater than a constant $\gamma>0$, the zeros of (4) are closer together than those of the equation

$$y_{xx}+gy=0, \quad g=\frac{\gamma}{P},$$

* For the special case of a transformation of the dependent variable only, this computation is given in § 5.

which has a solution $y = \sin \sqrt{g}(x+c)$ with zeros at intervals of $\frac{\pi}{\sqrt{g}}$. By fixing p, and taking G large enough in any interval of x, the zeros of (4) may then be made as close together as desired.

II. The special equation where $G(x, \lambda)$ contains the parameter linearly,

$$(py_x)_x + (q+\lambda k)y = 0, \tag{5}$$

has been much discussed. The boundary conditions

$$y(0) = y(1) = 0 \tag{6}$$

are of the greatest interest and three cases may be distinguished.

(A) *Orthogonal Case*, when $k(x) \geqq 0$. There is then an infinite number of parameter values $\lambda_m (\lambda_1 \leqq \lambda_2 \leqq \ldots .)$ with a limiting point at positive infinity only, for each of which a solution Y_m satisfying (6) exists. The number of zeros of the solution Y_m (including those at $x=0$ and $x=1$) is $m+1$.

(B) *Polar Case*, when $k(x)$ takes on both signs and the integral

$$D(y) = \int_0^1 (py_x^2 - qy^2)\, dx \tag{7}$$

is positive-definite,* that is, for the given boundary condition (6) $D(y)$ can not take on negative values. There are two sets each of an infinite number of parameter values $0 \leqq \lambda_1 \leqq \lambda_2 \ldots,\ 0 \geqq \lambda_{-1} \geqq \lambda_{-2} \ldots,$ with limiting points at positive and negative infinity respectively, corresponding to which solutions Y_m, Y_{-m} exist. For both Y_m and Y_{-m} the number of zeros is $m+1$.

(C) *Non-definite Case*, when both of the integrals

$$\int_0^1 ky^2 dx, \quad \int_0^1 (py_x^2 - qy^2)\, dx$$

may take on negative values. This will be discussed in § 4.

III. For the equation (5) and the special boundary conditions (6) certain minimum properties may be stated. In the orthogonal case the minimum of the integral $D(y)$ for those values of y which satisfy (6), and the normalizing and orthogonalizing conditions

$$\int_0^1 ky^2 dx = 1, \quad \int_0^1 kY_i y\, dx = 0, \qquad i = 1, 2, \ldots, m-1, \tag{8}$$

is λ_m, and is furnished by the normalized solution Y_m of (5). In the polar case the minimum of $D(y)$ for those values which satisfy conditions (6) and (8) is λ_m and is furnished by Y_m; the minimum subject to the conditions (6) and

$$\int_0^1 ky^2 dx = -1, \quad \int_0^1 kY_{-i} y\, dx = 0, \qquad i = 1, 2, \ldots, m-1,$$

* Bôcher has pointed out (*Proceedings International Congress of Mathematics, loc. cit.*, p. 173) that the special case of the polar problem where $q \leqq 0$ can be treated by the method of Sturm. This remark, however, does not apply to the polar case in its most general form.

is $-\lambda_{-m}$ and is furnished by Y_{-m}. In the non-definite case, $D(y)$ can be negative and the minimum (even in the simplest problem $(m=1)$) may not exist. However, in view of the developments of § 4, it would seem probable that by a modification of the discussion, the solution Y_m, for m large enough, may be regarded as furnishing a minimum of a calculus of variations problem.

IV. While the notion of regarding a differential equation directly as the limit of a set of difference equations has been used heuristically since the time of Euler, it was, so far as the author is aware, first made definite and rigorous in a recent paper.* In that paper the problem actually discussed is that of existence theorems for partial differential equations with given boundary conditions. But the same method applies, for example, to equation (1), and the discussion is essentially simpler for the equation in one dimension than for that in two or more. For the sake of simplicity let us confine ourselves to the case (5) (6) and consider the unit interval to be divided into m equal parts, the values of y, p, q, k at the point $\frac{i}{m}(i=0, 1, \ldots, m)$ to be denoted by y_i, p_i, q_i, k_i, and difference equations

$$m^2[p_{i+1}(y_{i+1}-y_i)-p_i(y_i-y_{i-1})]+q_iy_i+\lambda k_iy_i=0 \quad (i=1, 2, \ldots, m-1), \tag{9}$$

to be set up. In order that there be solutions of these equations λ must be one of the $m-1$ roots of the determinant formed from the coefficients. With increase of m the number of points at which y is defined increases, but we can pick out corresponding parameter values and solutions of the various sets of difference equations, and if proper continuity conditions are imposed on the coefficients of the differential equations, it may be shown that the corresponding sets of parameter values approach as a limit a parameter value of (5), and corresponding solutions approach a solution of (5). In this way the infinite set of solutions of the differential equation is obtained. If k has both signs and q is positive and sufficiently large, at least in some part of the interval, some of the λ's and the corresponding y's will be complex.† But in all those cases of equation (5) heretofore treated (the orthogonal and polar), the method of passing to the limit in (9) suffices and gives a simple proof of the fundamental facts. This method can be extended to a treatment of existence theorems for solutions of the equation (4) with more general boundary conditions.

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* *Transactions of the American Mathematical Society*, Vol. XVIII (1917), p. 489.

† One method of proving this would be by noting that the parameter values and solutions are approximations to those of the differential equation which are shown in § 4 to be complex.

§ 2. *Sufficient Conditions for the Existence of Solutions with an Unlimited Number of Zeros.*

In the consideration of the equation

$$(py_x)_x + G(x, \lambda) y = 0 \tag{10}$$

let us impose the restriction that for finite values of λ the function $G(x, \lambda)$ is limited, and in particular that $G(x, 0)$ is limited. The interesting cases will be covered by one of two hypotheses, which will be justified by the later developments of this section.

HYPOTHESIS A. *For at least one point of the interval, the upper limit of $G(x, \lambda)$ becomes infinite with λ ($\overline{\lim}_{\lambda=+\infty} G(x, \lambda) = +\infty$), and in such a manner that the number of zeros of the solutions increases without limit with λ.*

The problem treated by Birkhoff * where $\dfrac{\partial G}{\partial \lambda} > 0$ and $G(x, +\infty) = +\infty$ is a special case, and the orthogonal problem (§1, II) is still more special.

HYPOTHESIS B. *For at least one point of the interval $\overline{\lim}_{\lambda=+\infty} G(x, \lambda) = +\infty$; for at least one other, $\overline{\lim}_{\lambda=-\infty} G(x, \lambda) = +\infty$; and in both cases $G(x, \lambda)$ increases in such a manner that the number of zeros of the solutions increases without limit with λ.*

The polar case (§1, II) is included in this hypothesis.

THEOREM I. *In order that there be a set of parameter values λ such that the number of oscillations of the corresponding solutions of (10) be unlimited, it is necessary that in the neighborhood of at least one point $\overline{\lim}_{\lambda=+\infty} G(x, \lambda) = +\infty$ or $\overline{\lim}_{\lambda=-\infty} G(x, \lambda) = +\infty$.*

For, as the number of zeros increases, the length of the smallest interval decreases without limit. The zeros of the equation

$$Py_{xx} + G(x, \lambda) y = 0, \quad P = \text{maximum } p(x) \tag{11}$$

are farther apart than those of (10) (§ 1, I). To establish the theorem for (10) it is then only necessary to prove it for (11). Let us denote by α, β that pair of consecutive zeros of (11) whose distance is a minimum. In such an interval α, β [$y(\alpha) = y(\beta) = 0$], y may be taken positive, and since the equation is homogeneous, its solution for all values of λ may be multiplied by a constant so that the maximum is 1. The maximum of y_x must be at least as great as $\dfrac{1}{\beta-\alpha}$ which is the slope of the line joining $(\alpha, 0)$, $(\beta, 1)$. To investi-

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* *Loc. cit.*

gate the maximum of $-\frac{y_{xx}}{y}$ we note that y_x is zero at some point of the interval and at least as great as $\frac{1}{\beta-\alpha}$ at another. Hence $-y_{xx}$ must be at least as great as $\frac{1}{(\beta-\alpha)^2}$ at some point, and further $\max\left(-\frac{y_{xx}}{y}\right) \geqq \frac{1}{(\beta-\alpha)^2}$. Since for at least one sub-interval the length approaches zero, and since

$$G(x, \lambda) = -\frac{P y_{xx}}{y},$$

it follows that for the interval 0, 1 $\overline{\lim} \max G(x, \lambda) = +\infty$. Since by hypothesis G can become infinite only for $\lambda = \pm\infty$ the theorem may be readily deduced from these results by the usual processes of reasoning.

To show that the necessary condition of *Theorem* I is not sufficient, let us consider the following example: In the interval 0, $\frac{1}{2}-\varepsilon$ we set up the function $y = \frac{2}{\pi} \sin \frac{\pi x}{2}$, and in the interval $\frac{1}{2}+\varepsilon$, 1 the function $y = \frac{2}{\pi} \sin \frac{\pi}{2}(1-x)$; both of these arcs satisfy the equation $y_{xx} + \frac{\pi^2}{4} y = 0$. If continued, they would meet at the point $x = \frac{1}{2}$ with an angle arc tan $2\sqrt{2}$. If in the interval $\frac{1}{2}-\varepsilon$, $\frac{1}{2}+\varepsilon$ an analytic curve is introduced which is tangent to these two arcs $\left(\text{on one of which } y_x > \frac{1}{\sqrt{2}} \text{ and on the other } y_x < -\frac{1}{\sqrt{2}}\right)$, it must have at some point $-y_{xx} > \frac{1}{\sqrt{2}\varepsilon}$, and if ε is taken small enough, $\max\left(-\frac{y_{xx}}{y}\right)$ is great at pleasure. Hence denoting by $y_{xx} + G(x, \lambda) y = 0$ the differential equation which has for solution the function defined for the interval 0, 1 by this method, and setting $\lambda = \frac{1}{\varepsilon}$, we know that $\lim\limits_{\varepsilon=0} \max G(x, \lambda)$ in the neighborhood of $x = \frac{1}{2}$ increases without limit. On the other hand none of the suite of functions y has any zeros in the interval.

Having shown by this example that the necessary condition of *Theorem* I is not sufficient, let us now deduce a sufficient criterion that when $\lim\limits_{\lambda=\infty} G(x, \lambda) = \infty$, the number of zeros of solutions of (10) be unlimited. A similar discussion can be given for the case $\lim\limits_{\lambda=-\infty} G(x, \lambda) = \infty$. Denoting as before by P the maximum of $p(x)$, and by M an arbitrarily large constant, it is possible to find a λ and an x-interval such that $G(x, \lambda) > MP$ in this interval, whose length will be denoted by ε_M. The number of zeros of the solution $\sin \sqrt{M}(x+c)$ of the equation $y_{xx} + My = 0$ in an interval of length ε_M is not less than the integral

part of $\frac{\sqrt{M}\varepsilon_M}{\pi}$. It follows immediately from § 1, I that the number of zeros of solutions of (10) is not less than that of $\sin \sqrt{M}(x+c)$. Hence the

THEOREM II. *In order that there be N zeros in a solution of* (10) *it is sufficient that one can find an M such that $\frac{\sqrt{M}\varepsilon_M}{\pi} > N$, where ε_M is the length of a sub-interval where $G(x, \lambda) > MP$.*

COROLLARY I. *In order that the number of zeros be unlimited it is sufficient that $\lim\limits_{\lambda=\infty} \varepsilon_M \sqrt{M} = \infty$.*

COROLLARY II. *If throughout any sub-interval of fixed length the value of $G(x, \lambda)$ increases without limit, the number of zeros increases indefinitely.*

However, the sum of the sub-intervals in which $G > M$ may remain above a positive constant without compelling the number of zeros to increase with M. For example, one can set up a function which has no zeros except at $x=0$ and $x=1$ and which, except in the neighborhood of these points, oscillates between $y=\frac{1}{2}$ and $y=\frac{3}{2}$, the number of oscillations increasing indefinitely with λ. Moreover, by taking the oscillations frequent enough one can have $G(x, \lambda) > MP$ in portions which total at least one-quarter (or any other proper fraction) of the interval, the great curvature downward in these sub-intervals being counterbalanced by curvature upward in those where $G(x, \lambda)$ has the opposite sign.

On the other hand, we have seen that the number of zeros of the solution $\sin \sqrt{M}(x+c)$ of the equation $y_{xx} + My = 0$ is not less than the integral part of $\frac{\sqrt{M}\varepsilon_M}{\pi}$ where ε_M is the length of the interval. If this interval is divided into two parts ε_1, ε_2 for which there are solutions $\sin \sqrt{M}(x+c_1)$, $\sin \sqrt{M}(x+c_2)$ respectively, the number of zeros can not be reduced by more than one. For, the sum of the integral parts of $\frac{\sqrt{M}\varepsilon_1}{\pi}$ and $\frac{\sqrt{M}\varepsilon_2}{\pi}$ cannot differ by more than one from the integral part of $\frac{\sqrt{M}(\varepsilon_1+\varepsilon_2)}{\pi}$. It follows in the same way that if the interval is divided into n_1 parts the number of zeros of the solutions can not be decreased by more than n_1-1. But in the intervals where $G(x, \lambda) > MP$ the zeros of the solutions of (10) must be at least as many as the minimum number for $y_{xx} + My = 0$.

THEOREM III. *In order that the solution of* (10) *have N zeros it is sufficient that one can find an M such that $\frac{\sqrt{M}\eta_M}{\pi} - n_1 + 1 > N$, where η_M denotes the sum of the lengths of the n_1 intervals in which $\frac{G}{\max p(x)} > M$.*

§ 3. *Behavior of the Zeros with Monotone Change of* λ.

Let us fix the zero at the left-hand end of the interval 0, 1, denote such a solution by Y $[Y(0)=0]$, and investigate what happens to the other zeros as λ increases. It is convenient to think of $G(x, \lambda)$ as being defined for values $x>1$. Since the coefficients of (10) are continuous, the zeros move continuously, and since we can not have both $y(x)=0$ and $y'(x)=0$ without having $y\equiv 0$, the zeros can not coalesce and then disappear. By differentiating (10) with regard to λ we have

$$\frac{\partial(py_x)_x}{\partial\lambda}+G\frac{\partial y}{\partial\lambda}+\frac{\partial G}{\partial\lambda}y=0, \tag{12}$$

and on multiplication of this equation by $-y$ and of (10) by $\frac{\partial y}{\partial\lambda}$, addition and integration from x_1 to x_2 we get the fundamental formula *

$$\left[py_x\frac{\partial y}{\partial\lambda}\right]_{x_1}^{x_2}-\left[py\frac{\partial y_x}{\partial\lambda}\right]_{x_1}^{x_2}=\int_{x_1}^{x_2}\frac{\partial G}{\partial\lambda}y^2dx. \tag{13}$$

For the particular solution $Y(x)$ we have $Y(0)=0, \frac{\partial Y(0)}{\partial\lambda}$, and if α is another zero of Y the formula becomes

$$p(\alpha)Y_x(\alpha)\frac{\partial Y}{\partial\lambda}(\alpha)=\int_0^\alpha\frac{\partial G}{\partial\lambda}Y^2dx. \tag{13'}$$

If the integral on the right is positive, the signs of $Y_x(\alpha)$ and $\frac{\partial Y(\alpha)}{\partial\lambda}$ are the same. Hence when $Y_x(\alpha)$ is negative, $Y(\alpha)$ is decreasing with increase of λ, and this zero of Y is moving to the left; when $Y_x(\alpha)$ is positive, $Y(\alpha)$ is increasing and the zero is again moving to the left. In the same way it may it may be shown that all zeros for which the integral on the right of (13′) is negative, move to the right. When the integral is zero, further investigation is needed. We have now proved

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* If p is also a function of λ this formula becomes

$$\left[py_x\frac{\partial y}{\partial\lambda}\right]_{x_1}^{x_2}-\left[py\frac{\partial y_x}{\partial\lambda}\right]_{x_1}^{x_2}-\left[\frac{\partial p}{\partial\lambda}y_xy\right]_{x_1}^{x_2}=\int_{x_1}^{x_2}\frac{\partial G}{\partial\lambda}y^2dx-\int_{x_1}^{x_2}\frac{\partial p}{\partial\lambda}y^2_xdx.$$

In terms of this hypothesis results analogous to all the succeeding theorems of this paper may be written down. Since, however, by a transformation of variables p can be made independent of λ we shall, for the sake of simplicity, confine ourselves to the problem proposed.

If at $x=0$ the condition $y_x(0)+\sigma y(0)=0$ is imposed where σ is a constant, then $\frac{\partial y_x(0)}{\partial\lambda}+\sigma\frac{\partial y(0)}{\partial\lambda}=0$, and this formula (13) can be written

$$-y^2p\frac{d}{dx}\left(\frac{\frac{\partial y}{\partial\lambda}}{y}\right)\equiv p\left[y_x\frac{\partial y}{\partial\lambda}-y\frac{\partial y_x}{\partial\lambda}\right]=\int_0^x\frac{\partial G}{\partial\lambda}y^2dx.$$

Hence, in order that the roots of a solution y and the roots of $\frac{\partial y}{\partial\lambda}$ alternate it is sufficient that $\int_0^x\frac{\partial G}{\partial\lambda}y^2dx$ have one sign.

THEOREM IV. *If $Y(0)=0$ and if λ is a parameter value for which there are subsequent zeros of Y at $\alpha_1, \ldots, \alpha_n$ then with increase of λ a zero, α_i, moves to the left or right according as*

$$\int_0^{\alpha_i} \frac{\partial G(x,\lambda)}{\partial \lambda} Y^2(x,\lambda)\,dx$$

is positive or negative.

The argument used in deducing *Theorem* IV is still valid if in place of a zero of y we introduce a zero of y_x.

THEOREM IV A. *If $y(0)=0$ or $y_x(0)=0$ and if λ is a parameter value such that $y(\alpha)=0$ or $y_x(\alpha)=0$, then with increase of λ the zero α moves to the left or right according as*

$$\int_0^{\alpha} \frac{\partial G(x,\lambda)}{\partial \lambda} y^2(x,\lambda)\,dx$$

is positive or negative.

COROLLARY. *In order that λ be a multiple characteristic number and $Y(x,\lambda)$ a multiple solution of the problem* (10) (6) *it is necessary** *that*

$$\int_0^{1} \frac{\partial G(x,\lambda)}{\partial \lambda} Y^2(x,\lambda)\,dx=0.$$

G can be so chosen as a function of λ that with increasing λ the integral $\int_0^1 \frac{\partial G}{\partial \lambda} Y^2 dx$ is alternately positive and negative, and since the corresponding intervals of λ may be small at pleasure, we see that a zero may pass back and forth through $x=1$ an infinite number of times. There may then be an infinite number of λ's for which exist solutions of (10) (6) with a fixed number of zeros.

As λ runs through its interval there will be at least one value for which the number of zeros of $Y(x,\lambda)$ is a minimum. For both the orthogonal and polar cases this minimum number of oscillations is zero, but in general this will not be the case. If for $\lambda=\lambda'$ there is a minimum number n_1 of oscillations, then from the principle of continuity we can argue that for $n>n_1$ there is under *Hypothesis A* (§ 2) at least one value of λ for which exists a solution of (10) vanishing at $x=0$ and $x=1$, and oscillating n times (having n zeros) in the interval, while under *Hypothesis B* there are at least two.

One fundamental problem for investigation is the determination of sufficient conditions that in a given interval of λ (which may include infinity) there is *only one* solution of (10) (6) oscillating a given number of times; in

* This condition may be shown to be sufficient also.

other words, sufficient conditions that the zeros of $Y(x)$ pass through the point $x=1$ in one direction only as λ increases (or decreases). Another problem is that of determining when there exist precisely two solutions oscillating a given number of times.

THEOREM V. *If for values of λ in an interval (which may include infinity) there are solutions of* (10) (6) *for which the minimum and maximum number of oscillations are n_1 and n_2 (n_2 may be infinite), respectively, and if the value of $\frac{\partial G}{\partial \lambda}$ is equal to or greater than $\phi(\lambda) G(x, \lambda)$, where ϕ is a positive function of λ, then there is one and only one solution oscillating n times ($n_1 \leqq n \leqq n_2$) in the interval $0 \leqq x \leqq 1$.*

For, we have on multiplying (10) by Y, and integrating under the boundary conditions (6),

$$\int_0^1 p Y_x^2 dx = \int_0^1 G Y^2 dx. \tag{14}$$

From the hypothesis we may then write

$$\int_0^1 \frac{\partial G}{\partial \lambda} Y^2 dx \geqq \phi(\lambda) \int_0^1 G Y^2 dx = \phi(\lambda) \int_0^1 p Y_x^2 dx > 0$$

and since this holds for all λ, *Theorem* IV gives the desired result.

The theorem just proved fits the case where with unlimited increase of λ the maximum of G increases without limit. For the case where λ approaches $-\infty$, we observe from *Theorem* IV that a sufficient condition for a similar theorem is that $\int_0^{\prime} \frac{\partial G}{\partial \lambda} Y^2 dx$ be negative. To insure this, it is sufficient to take $\frac{\partial G}{\partial \lambda} < -\psi G$, where $\psi(\lambda)$ is a positive function. Then

$$\int \frac{\partial G}{\partial \lambda} Y^2 dx < -\psi(\lambda) \int G Y^2 dx = -\psi(\lambda) \int p Y_x^2 dx < 0.$$

Combining this result with *Theorem* V we may state the following:

THEOREM VI. *If for two values λ', λ'' ($\lambda' \geqq \lambda''$) of λ there are n_1 zeros of the solution Y within the interval* 0, 1, *and if for $\lambda > \lambda'$, $\frac{\partial G}{\partial \lambda} \geqq \phi_1(\lambda) G(x, \lambda)$; and for $\lambda < \lambda''$, $\frac{\partial G}{\partial \lambda} \leqq -\phi_2(\lambda) G(x, \lambda)$ where ϕ_1, ϕ_2 are positive functions defined in the intervals λ', $+\infty$ and $-\infty$, λ'', respectively, then under Hypothesis B* (§ 2) *for $n \geqq n_1$ there are two and only two values of λ for which there is a solution $Y(x, \lambda)$ of* (10) (6) *with n zeros.*

In the polar problem we have the special case of the hypothesis of this theorem where $\lambda'=\lambda''=0$, $n_1=0$, $G=q+\lambda k$, $\dfrac{\partial G}{\partial\lambda}=k$, $\phi_1=\dfrac{1}{\lambda}>0$, $-\phi_2=\dfrac{1}{\lambda}<0$. A more general special case of *Theorem* VI is discussed in § 4.

Since formula (14) is valid also when

$$y(0)=y_x(1)=0, \text{ or } y_x(0)=y(1)=0, \text{ or } y_x(0)=y_x(1)=0,$$

theorems analogous to VI can be written down for any of these boundary conditions.

§ 4. *The Non-Definite Case.*

With regard to applications, the differential equation

$$(py_x)_x+(q+\lambda k)y=0, \tag{15}$$

with boundary conditions

$$y(0)=y(1)=0, \tag{16}$$

is the most important special type of (10). The problem where one of the integrals

$$D(y,0,1)=\int_0^1(py_x^2-qy^2)\,dx,\quad \int_0^1 ky^2dx$$

is definite, has been studied in great detail. Let us first investigate some properties of the remaining *non-definite case* where both these integrals may take on negative values and later deduce some analogous results for the more general equation (10). In §§ 6–8 these results will be extended to cover the more general boundary condition (2).

For $q\leqq 0$ the integral $D(y,0,1)$ can not be negative. If q is positive in at least a part of the interval 0, 1, the minimum of the integral $\int_0^1 py_x^2dx$ under the conditions (16) and $\int_0^1 qy^2dx=1$ is (§ 1, III) the smallest parameter value λ of the equation $(py_x)_x+\lambda qy=0$, and when $\dfrac{q}{\min p}\leqq\pi^2$, this in turn is (§ 1, I) at least as great as the smallest of the equation $y_{xx}+\lambda\pi^2y=0$ under the same boundary conditions (16). Since for the latter equation $\lambda_1=1$ and $Y_1=c\sin\pi x$, it follows that λ is greater than unity. Hence $\int_0^1 py_x^2dx=\lambda$ is greater than $\int_0^1 qy^2dx=1$ and $D(y,0,1)$ is positive.

In the non-definite case one must therefore have $\dfrac{q}{\min p}>\pi^2$ *in at least a part of the interval*, and this assumption we shall now make. But we note here that in a sub-interval α_1, α_2 such that $y(\alpha_1)=y(\alpha_2)=0$, any value of q for which $\dfrac{q}{\min p}<\dfrac{\pi^2}{(\alpha_2-\alpha_1)^2}$, makes the integral $\int_{\alpha_1}^{\alpha_2}(py_x^2-qy^2)\,dx$ definite, as is

readily shown by a comparison argument like that above. Hence, for given p and q, if throughout the interval 0, 1 the zeros can be taken close enough together (as *e. g.* may certainly be done where λ has the same sign as $k(x)$ and is taken large enough in absolute value, cf. § 1, I), the integral for each sub-interval is definite, and hence $D(y, 0, 1)$ is positive. *This is also true under the more general boundary conditions* (2).

On the other hand from the formula

$$D(y, \alpha_1, \alpha_2) \equiv \int_{\alpha_1}^{\alpha_2} (py_x^2 - qy^2)\, dx = \lambda \int_{\alpha_1}^{\alpha_2} ky^2 dx, \tag{17}$$

which is obtained by multiplying (15) by y and integrating under the conditions $y(\alpha_1) = y(\alpha_2) = 0$, we see that if λ is positive and k negative in the interval, then $D(y, \alpha_1, \alpha_2)$ is negative; it is, however, still possible that for the larger interval 0, 1 the integral $D(y, 0, 1)$ be positive. Were this the case (as will later be established provided λ is large enough) we could deduce from *Theorem IV* and the formula

$$D(y, 0, 1) \equiv \int_0^1 (py_x^2 - qy^2)\, dx = \lambda \int_0^1 ky^2 dx, \quad y(0) = y(1) = 0, \tag{18}$$

certain facts in regard to the movement of the subsequent zeros of particular solutions Y of (15) which vanish at $x=0$. Before proceeding to a detailed discussion of this matter let us compare some possibilities for the orthogonal, polar and non-definite cases.

In the orthogonal case let us consider all λ greater than some finite number chosen less than the smallest characteristic (which may be positive or negative). As λ increases, the subsequent zeros of any solution Y of (15) $[Y(0) = 0]$ move to the left, new ones being added to the interval 0, 1. For the smallest value of λ there are no zeros within the interval and when λ passes through a parameter value λ_n the n-th zero enters. In the polar case there are two ranges of values of λ; for the range which extends from zero to $+\infty$, a result precisely like that of the orthogonal case may be stated; for the range which extends from $-\infty$ to zero, a decrease of λ causes subsequent zeros to move to the left, there being no zero of $Y(x, 0)$ present in the interval while the n-th enters for $\lambda = \lambda_{-n}$. In the non-definite case we know that the results must be quite different and we may expect that the march of the zeros will not be monotone with λ. In fact, *there may be a range of values of* λ $(L_1 \leqq \lambda \leqq L_2)$ *such that as* λ *increases the number of zeros first decreases, then increases, then decreases and finally increases, the minimum number being a positive integer.**

* A reference to the proof of this last fact is given in the Introduction.

As an example let us consider the equation

$$y_{xx}+[(100\pi)^2+\lambda\varkappa)y=0, \quad y(0)=y(1)=0,$$

where $\varkappa$ is equal -1 in the sub-interval $0, \frac{1}{2}$ and equal 1 in the remainder of the interval. Such a function $\varkappa(x)$ may be approximated by an analytic function $k(x)$ for which the corresponding equation has solutions with similar properties. When $\lambda=0$ the solution $y=\sin 100\pi x$ has 100 zeros in the interval. When $\lambda=(100\pi)^2$, we have the equation $y_{xx}=0$ in $0, \frac{1}{2}$ and $y_{xx}+2(100\pi)^2y=0$ elsewhere, the solutions being respectively $y=cx$ ($c=$const.), $y=\sin 100\sqrt{2}\pi x$. There are are then approximately $50\sqrt{2}$ zeros. For $\lambda=-(100\pi)^2$ there are evidently the same number of zeros. More generally, when λ is in the interval $-(100\pi)^2, (100\pi)^2$ the number of zeros is approximately

$$\frac{1}{2\pi}[\sqrt{(100\pi)^2+\lambda}+\sqrt{(100\pi)^2-\lambda}],$$

a function which has its maximum for $\lambda=0$ and *decreases* with increase of $|\lambda|$. On the other hand when $|\lambda|$ is greater than $(100\pi)^2$ the number of zeros is approximately $\frac{1}{2\pi}\sqrt{(100\pi)^2+|\lambda|}$, which *increases* with $|\lambda|$. The minimum number of zeros is then approximately $50\sqrt{2}$ and occurs for $\lambda=(100\pi)^2$ and $-(100\pi)^2$.* *For values of n between* $50\sqrt{2}$ *and* 100 *there are four parameters corresponding, to which exist solutions oscillating n times.*

Returning now to the discussion of the sign of $D(y, 0, 1)$ we can prove

LEMMA I. *If in an interval a, b, the function k is positive, except perhaps at the end points, then λ can be taken so large that for solutions of* (15),

$$D(y, a, b)\equiv\int_a^b(py_x^2-qy^2)\,dx>0.$$

For, by taking λ large enough, $q+\lambda k$ can be made as large as desired, except perhaps at the end points. Hence the zeros are as thickly strewn throughout the interval a, b as desired (§ 1, I). We note that in any sub-interval β_1, β_2 of a, b, in which there is a zero of y and for which the maximum of $|y|$ is M, the minimum value of $\int_{\beta_1}^{\beta_2} y_x^2dx$ is taken on when y is a straight line joining $(\beta_1, 0)$ and $(\beta_2, \pm M)$. This minimum value is $\frac{M^2}{\beta_2-\beta_1}$. Since on the other hand $\int_{\beta_1}^{\beta_2} qy^2dx<(\beta_2-\beta_1)M^2 \max q$, the value of $D(y, \beta_1, \beta_2)$ is greater than $M^2\left[\frac{\min p}{\beta_2-\beta_1}-(\beta_2-\beta_1)\max q\right]$: hence when $(\beta_2-\beta_1)^2$ is made less than

* By reference to *Theorem* IV it follows from this discussion that in the two intervals which are approximately $-(100\pi)^2, 0$; $(100\pi)^2, +\infty$ the value of $\int_0^1 \varkappa y^2dx$ must be positive or $\int_0^{\frac{1}{2}} y^2dx<\int_{\frac{1}{2}}^1 y^2dx$, while in the intervals which are approximately $-\infty, -(100\pi)^2$; $0, (100\pi)^2$, $\int_0^{\frac{1}{2}} y^2dx>\int^1 y^2dx$.

$\frac{\min p}{\max q}$ this integral is definite. It follows that since the first and last zeros of $y(x)$ are as close to the ends of the interval a, b as is desired, those portions of the integral arising from these two end sub-intervals can be made positive. For any interval α_1, α_2 of oscillation $[y(\alpha_1)=y(\alpha_2)=0]$ it follows at once from (17) that when k and λ are positive $D(y, \alpha_1, \alpha_2)>0$. Combining these results we have the lemma.

It follows in the same way that if $k \leqq 0$ and $k \not\equiv 0$ in an interval a, b, then λ can be taken so large negatively that for solutions of (15) the integral $D(y, a, b)$ is positive.

Coming now to an interval c, d in which k is negative, except perhaps at c and d, it is possible in an interval $c+\eta, d-\eta$ ($\eta>0$ arbitrarily small) to take M' arbitrarily large and then choose λ so large that $-(q+\lambda k)>M'$ or

$$|(py_x)_x|=|-(q+\lambda k)y|>M'|y|, \quad [(py_x)_x>M'y \text{ if } y>0]. \tag{19}$$

The arc $y(x)$ may be shown to be sharply concave away from the x-axis.* It is possible to prove the following:

Lemma II. *If in an interval c, d the function k is negative, except perhaps at the end points, then λ can be taken so large that for a solution of* (15)

$$D(y, c, d) \equiv \int_c^d (py_x^2 - qy^2)\,dx > 0.$$

This will be proved by showing that on taking λ great enough it is possible to insure that $\frac{y_x^2}{y^2}$ is great at pleasure except perhaps in sub-intervals whose length decreases indefinitely with $\frac{1}{\lambda}$. Since in the interval $c+\eta, d-\eta$ the curve is concave upward for positive y and concave downward for negative y, there can not be more than one zero. The discussion may be separated into two parts according as $y(x)$ has a zero or not.

If $y(\gamma)=0$, then without loss of generality it may be assumed that within the interval $\gamma, d-\eta$, y is positive and hence, since $y_{xx}>0$, that y_x is positive. Multiplying the inequality (19) by $2py_x$ and integrating we get

$$\int_\gamma^x 2py_x(py_x)_x dx > M' \int_\gamma^x 2ypy_x dx,$$

and hence

$$p^2(x)y_x^2(x) \geqq p^2(x)y_x^2(x) - p^2(\gamma)y_x^2(\gamma) \geqq M'[\min p]\int_\gamma^x 2yy_x dx$$
$$= M'y^2(x)\min p, \tag{20}$$

---

* This is at once evident if the equation is taken in the form $\bar{y}_{xx}+(\bar{q}+\lambda\bar{k})\bar{y}=0$ to which it may be reduced by a transformation.

for all x in the interval γ, $d-\eta$. Similar reasoning establishes the same result in the interval $c+\eta$, γ. Hence, for the case that y vanishes, we have throughout the interval $c+\eta$, $d-\eta$ the inequality $\frac{y_x^2}{y^2} > \frac{M' \min p}{p^2}$, which as noted above, is sufficient to establish the theorem.

If on the other hand y has no zero in the interval, let us take the function positive and denote by γ the point at which it takes on its minimum. It may be assumed that γ is not at the right-hand end of the interval; that special case could be treated in an analogous fashion. By multiplying y by a constant, $y(\gamma)$ can be made equal 1, while the sign of the integral $D(y, c, d)$ is not altered. Since $y_x(\gamma) \geqq 0$, we have $y > 1$ in the neighborhood of γ; in fact, for any arbitrarily small but fixed η we can, by taking M' large enough, have $y(\gamma+\eta) > 2$. For, since $y \geqq 1$ it follows from the inequality (19) that

$$py_x \geqq py_x - py_x(\gamma) = \int_\gamma^x (py_x)_x dx \geqq M' \int_\gamma^x y dx = M'(x-\gamma).$$

Integrating again,

$$(y-1) \max p = \max p \int_\gamma^x y_x dx \geqq \int_\gamma^x py_x dx \geqq M' \int_\gamma^x (x-\gamma)\, dx = \frac{M'(x-\gamma)^2}{2}.$$

By taking $x-\gamma \geqq \eta$ and choosing $\frac{M'\eta^2}{2 \max p} > 1$ we have $y-1>1$ or $y>2$. From this it follows that on multiplying the inequality (19) by $2py_x$ and integrating we get by a process similar to that used in (20) the formula

$$p^2(x)\, y_x^2(x) > M' \int_\gamma^x 2ypy_x dx \geqq M'[y^2(x)-1] \min p > \frac{3M'}{4} y^2(x) \min p.$$

Hence in the interval $\gamma+\eta$, $d-\eta$ we have $\frac{y_x^2}{y^2} > \frac{3M'}{4} \frac{\min p}{p^2}$. And since in the interval $c+\eta$, $\gamma-\eta$ a similar inequality may be obtained, this completes the proof of the lemma.

It follows in the same way that if $k>0$, except perhaps at the end points of the interval c, d, then λ can be taken so large negatively that for solutions of (15) the integral $D(y, c, d)$ is positive.

We have then proved by these lemmas that in any sub-interval in which k has one sign, $D(y)$ can be made positive by taking λ large enough positively and also by taking λ large enough negatively. It follows that $D(y, 0, 1)$ would be positive, and from the formula

$$\lambda \int_0^1 ky^2 dx = \int_0^1 (py_x^2 - qy^2)\, dx \equiv D(y, 0, 1) > 0$$

and *Theorem* IV, that for $|\lambda|$ large enough the zeros of Y move to the left with increase of $|\lambda|$. We can then enunciate

THEOREM VII. *There exists an integer n_2 such that for $n \geqq n_2$ there are precisely two solutions of* (15) (16) *oscillating n times.*

If proper restrictions are imposed on $G(x, \lambda)$, a theorem similar to the preceding may be proved for the general equation

$$(py_x)_x + G(x, \lambda) y = 0. \tag{10}$$

The proof used in *Theorems* VI and VII, and *Lemmas* I and II is valid also under the following hypotheses on a function $G'(x, \lambda)$ obtained by subtracting from $G(x, \lambda)$ a function $q(x) \cdot [G'(x, \lambda) \equiv G(x, \lambda) - q(x)]$.

(α) In one or more sub-intervals of 0, 1, $G'(x, \lambda)$ is a monotone increasing function of λ such that $\lim\limits_{\lambda=\infty} G'(x, \lambda) \geqq 0$ and for at least a part of each sub-interval $\lim\limits_{\lambda=\infty} G(x, \lambda) = +\infty$.

(β) In the remaining sub-interval or sub-intervals of 0, 1, $G'(x, \lambda)$ is a decreasing function of λ such that $\lim\limits_{\lambda=-\infty} G'(x, \lambda) \leqq 0$, and for at least a part of each sub-interval $\lim\limits_{\lambda=-\infty} G'(x, \lambda) = -\infty$.

(γ) For values λ', λ'' of λ there are n_1 zeros of the solution Y within the interval 0, 1 and for $\lambda > \lambda'$,

$$\frac{\partial G'}{\partial \lambda} \geqq \phi_1(\lambda) G'(x, \lambda), \text{ and for } \lambda < \lambda'', \frac{\partial G'}{\partial \lambda} \leqq -\phi_2(\lambda) G'(x, \lambda),$$

where ϕ_1, ϕ_2 are positive functions defined in the intervals λ', $+\infty$ and $-\infty$, λ'' respectively.

THEOREM VIII. *Under the conditions* (α) (β) (γ) *there exists an integer n_2 such that for $n > n_2$ there are precisely two solutions of* (10) (16) *oscillating n times.*

When the parameter value and corresponding solution of (15) (16) are complex, let us set $\lambda \equiv \sigma + i\tau$, $y \equiv u + iv$. The differential equation resolves itself into the two following:

$$(pu_x)_x + qu + \sigma ku - \tau kv = 0, \quad u(0) = u(1) = 0, \tag{21}$$

$$(pv_x)_x + qv + \sigma kv + \tau ku = 0, \quad v(0) = v(1) = 0. \tag{22}$$

Corresponding to $\lambda \equiv \sigma - i\tau$ there is then a solution $y \equiv u - iv$. On multiplying (21) and (22) in the first place by u and v, and in the second by v and $-u$ respectively, adding and integrating, one obtains the two formulae

$$\int_0^1 [p(u_x^2 + v_x^2) - q(u^2 + v^2)]\, dx = 0, \quad \int_0^1 k(u^2 + v^2)\, dx = 0. \tag{23}$$

More generally one obtains by this process the formula

$$[p(u_x v - v_x u)]_{x=\alpha}^{x=\beta} = \tau \int_\alpha^\beta k(u^2+v^2)\,dx. \tag{24}$$

The orthogonality of two solutions ($\int_0^1 kY_n Y_m dx = 0$, $m \neq n$) holds as well for complex as for real characteristic numbers λ_n, λ_m. Hence by separation into real and imaginary parts we get the

THEOREM IX. *If* $Y_m(x, \lambda_m) \equiv u_m + iv_m$, $Y_n \equiv u_n + iv_n$ *are two solutions of* (15) (16), *then*

$$\int_0^1 k(u_m u_n - v_m v_n)\,dx = 0, \quad \int_0^1 k(u_m v_n + u_n v_m)\,dx = 0.$$

Theorems of other types may be derived of which the following is an example:

THEOREM X. *If in the interval* 0, 1, *the function* $k(x)$ *changes sign once only, then the roots of the real and imaginary parts* u_n, v_n *of the solution* $Y_n \equiv u_n + iv_n$ *of* (15) (16) *separate one another.*

For, formula (24) may be written

$$p(u_x v - v_x u) = \int_0^x k(u^2+v^2)\,dx, \text{ or } p\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{1}{v^2}\int_0^x k(u^2+v^2)\,dx.$$

From the hypothesis that k changes sign but once it follows that the integral can not vanish within the interval, and since $\dfrac{d\left(\frac{u}{v}\right)}{dx} > 0$ except at the end points, $\dfrac{u}{v}$ is a monotone function and the theorem follows at once. We may note further that under the hypotheses of the theorem neither u nor v can gain or lose a zero at a point within the interval. For, were there a zero lost or gained at $x = a$, both $y(a)$ and $y_x(a)$ would vanish, and formula (24) could be written $\int_0^a k(u^2+v^2)\,dx = 0$, which would give a contradiction.

§ 5. *Reduction of the General Boundary Conditions to Normal Forms.*

In the preceding sections the discussion has dealt mainly with the simple boundary conditions $y(0)y_x(0) = y(1)y_x(1) = 0$. To facilitate the discussion of the most general boundary conditions it is desirable to obtain normal forms.* The linear self-adjoint equation of the second order will be taken in the form

$$(\pi(x)u_x(x))_x + \Gamma(x, \lambda)u(x) = 0, \quad \pi(x) > 0, \tag{25}$$

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* The classification used in this section follows that of Hilbert and Haupt, *loc. cit.* The geometrical form into which the transformation is thrown was suggested by my colleague, Prof. H. P. Manning.

and if the linearly independent boundary conditions

$$\left.\begin{aligned}\alpha_1 u(0)+\alpha_2 u_x(0)+\alpha_3 u(1)+\alpha_4 u_x(1)=0,\\ \beta_1 u(0)+\beta_2 u_x(0)+\beta_3 u(1)+\beta_4 u_x(1)=0,\end{aligned}\right\} \tag{26}$$

are to be self-adjoint, the condition

$$\pi(0)\begin{vmatrix}\bar{u}(0) & \bar{u}_x(0)\\ \bar{\bar{u}}(0) & \bar{\bar{u}}_x(0)\end{vmatrix}=\pi(1)\begin{vmatrix}\bar{u}(1) & \bar{u}_x(1)\\ \bar{\bar{u}}(1) & \bar{\bar{u}}_x(1)\end{vmatrix} \tag{27}$$

must be imposed, where $\bar{u}$, $\bar{\bar{u}}$ are any functions with continuous derivatives which satisfy (26). We have the relation

$$\begin{vmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\end{vmatrix}\begin{vmatrix}\bar{u}(0) & \bar{u}_x(0)\\ \bar{\bar{u}}(0) & \bar{\bar{u}}_x(0)\end{vmatrix}=\begin{vmatrix}\alpha_3 & \alpha_4\\ \beta_3 & \beta_4\end{vmatrix}\begin{vmatrix}\bar{u}(1) & \bar{u}_x(1)\\ \bar{\bar{u}}(1) & \bar{\bar{u}}_x(1)\end{vmatrix}, \tag{28}$$

as may be seen by applying the usual rule for multiplying determinants and noting from (26) that each of the four elements of one product determinant is the negative of the corresponding element of the other. Let us denote by B_{ij} the determinant formed by taking the i-th and j-th columns of the matrix

$$\begin{Vmatrix}\alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \beta_1 & \beta_2 & \beta_3 & \beta_4\end{Vmatrix}.$$

Considering the two determinants of the *u's* as the variables, we have from the theory of linear equations that a necessary and sufficient condition for the solution of (27) and (28) is that

$$\pi(1)B_{12}-\pi(0)B_{34}=0. \tag{29}$$

Hence B_{12} and B_{34} are simultaneously zero or different from zero.

Let us subject the dependent variable to the transformation

$$u(x)=\eta(x)y(x),\quad \eta(x)\neq 0, \tag{30}$$

by which the number of zeros of the solution remains unaltered. The equation (25) takes on the self-adjoint form

$$(py_x)_x+G(x,\lambda)y=0,\quad p=\pi\eta^2>0,\quad G(x,\lambda)=\pi\eta\eta_{xx}+\pi_x\eta\eta_x+\Gamma\eta^2, \tag{31}$$

and the conditions (26) are replaced by a similar set

$$\left.\begin{aligned}\gamma_1 y(0)+\gamma_2 y_x(0)+\gamma_3 y(1)+\gamma_4 y_x(1)=0,\\ \delta_1 y(0)+\delta_2 y_x(0)+\delta_3 y(1)+\delta_4 y_x(1)=0.\end{aligned}\right\} \tag{32}$$

We have at once from (31) the important formula

$$\int_0^1 \frac{\partial\Gamma}{\partial\lambda}u^2dx=\int_0^1\frac{\partial G}{\partial\lambda}y^2dx. \tag{33}$$

Writing u_1, u_2, u_3, u_4, for $u(0)$, $u_x(0)$, $u(1)$, $u_x(1)$, and using a similar notation for the η's and y's we can interpret our problem in relation to the tetraedron of reference as the investigation of the line (26) subject to the condition (27) or (29). If $B_{34}=0$, the u_3 and u_4 can be eliminated from (26) at the same time, giving

$$B_{13}u_1+B_{23}u_2=0 \quad \text{or} \quad B_{14}u_1-B_{42}u_2=0, \tag{34}$$

either of these equations representing the plane determined by the given line and the edge $12(u_1=0,\ u_2=0)$. Hence the line (26) intersects this edge of the tetraedron. But if $B_{34}=0$ we have from (29) $B_{12}=0$, and the straight line intersects also the edge 34. The equation of the plane determined by the given line and the edge 34 can be written

$$B_{13}u_3+B_{14}u_2=0 \quad \text{or} \quad B_{23}u_3-B_{42}u_2=0. \tag{35}$$

From (34) and (35) we see that under the hypothesis $B_{12}=B_{34}=0$ the boundary conditions may be written

Case I. $$\sigma u_1+u_2=0, \quad \tau u_3+u_4=0, \tag{36}$$

where

$$\sigma=\frac{B_{13}}{B_{23}}=-\frac{B_{14}}{B_{42}}, \quad \tau=\frac{B_{13}}{B_{14}}=-\frac{B_{23}}{B_{42}}.$$

The parameters σ and τ may have zero or infinite values when the line (26) lies in a face of the tetraedron.

In general, when any two B's with complementary indices are zero, the line intersects two opposite edges of the tetraedron and the conditions (26) may be reduced to a normal form similar to (36).

The transformation (30) takes the form

$$u_1=\eta_1 y_1, \quad u_2=\eta_2 y_1+\eta_1 y_2, \quad u_3=\eta_3 y_3, \quad u_4=\eta_4 y_3+\eta_3 y_4, \quad \eta_1\neq 0,\ \eta_2\neq 0, \tag{37}$$

and corresponds to a rotation of the face 2 of the tetraedron about the edge 12, and of the face 4 about the edge 34, leaving the faces 1 and 2 and the edges 12, 13 and 34 unchanged. If we denote by A_{ij} the determinants of the matrix of the coefficients γ_i, δ_i of (32) we can read their values from the identity

$$\left|\begin{matrix} \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{matrix}\right| = \left|\begin{matrix} \eta_1\alpha_1+\eta_2\alpha_2 & \eta_1\alpha_2 & \eta_3\alpha_3+\eta_4\alpha_4 & \eta_3\alpha_4 \\ \eta_1\beta_1+\eta_2\beta_2 & \eta_1\beta_2 & \eta_3\beta_3+\eta_4\beta_4 & \eta_3\beta_4 \end{matrix}\right|;$$

thus $A_{12}=\eta_1^2 B_{12}$, $A_{34}=\eta_3^2 B_{34}$, etc. As may be seen from these formulae and (31), we have corresponding to (27) or (29), the relation

$$p(1)A_{12}-p(0)A_{34}=0 \tag{38}$$

which, when added to (32), makes these boundary conditions self-adjoint.

The discussion of the boundary conditions under the transformation may be sub-divided as follows:

I. If the given line intersects the edges 12 and 34 ($B_{34}=B_{12}=0$), the same will be true after the transformation since $A_{34}=A_{12}=0$. To reduce the condition to the normal form (36) (which we shall in general use) the transformation is superfluous. This case is called the sturmian.

From the formulae (37) it follows at once that the conditions (36) can by proper choice of $\eta_1, \eta_2, \eta_3, \eta_4$ be reduced to one of the simpler forms

$$\text{(a) } y_1=y_3=0; \quad \text{(b) } y_1=y_4=0; \quad \text{(c) } y_2=y_3=0; \quad \text{(d) } y_2=y_4=0, \tag{39}$$

these corresponding geometrically to the cases where the given line coincides respectively with the edge 13, the new edge 14, the new edge 23 or the new edge 24.

II. If the line does not intersect the edges 12, 34, but does intersect 13, the transformation may be so determined that it will also intersect 24; *e. g.*, by making the plane 2 pass through the intersection of the given line with the plane 4. In this case since u_4 is unaltered we have $\eta_3=1$, $\eta_4=0$, and hence to equate $A_{13}\equiv\eta_3(\eta_1 B_{13}+\eta_2 B_{23})$ to zero we need only to choose $\frac{\eta_2}{\eta_1}=-\frac{B_{13}}{B_{23}}$. After the transformation we have $A_{13}=A_{24}=0$, and by elimination in (32) we get by aid of (38)

Case II. $$y_1=hy_3, \quad hp(0)y_2=p(1)y_4,$$

where

$$h=\frac{A_{23}}{A_{12}}=-\frac{A_{34}}{A_{14}}=-\frac{p(1)A_{12}}{p(0)A_{14}}=\frac{p(1)A_{23}}{p(0)A_{34}}.$$

III. If the line does not intersect any of the three edges 12, 34, 13, ($B_{34}\neq 0$, $B_{12}\neq 0$, $B_{24}\neq 0$) we can determine the transformation so that the face 2 of the tetraedron shall pass through the intersection of the line and the face 3, while the face 4 passes through the intersection of the line and the face 1. Thus, in the new tetraedron, the line intersects the edges 23 and 14, and we have the relations $A_{14}\equiv\eta_3(\eta_1 B_{14}+\eta_2 B_{24})=0$, $A_{23}\equiv\eta_1(\eta_3 B_{23}+\eta_4 B_{24})=0$, which determine the ratios $\frac{\eta_1}{\eta_2}$, $\frac{\eta_3}{\eta_4}$ desired in this case. The equations of the line can now be written

Case III. $$y_1=lp(1)y_4, \quad lp(0)y_2=-y_3,$$

where

$$l=-\frac{A_{42}}{p(1)A_{12}}=\frac{A_{34}}{p(1)A_{13}}=\frac{A_{12}}{p(0)A_{13}}=-\frac{A_{42}}{p(0)A_{34}}.$$

THEOREM XI. *By a change of dependent variable the self-adjoint equation* (25) *with self-adjoint boundary conditions* (26) (27) *may be reduced to an equation* (31) *which is again self-adjoint, and for which the self-adjoint boundary condition may be written in one of three forms characterized respectively by* $A_{12}=A_{34}=0$; $A_{23}=A_{14}=0$; $A_{13}=A_{42}=0$:

Case I. $\sigma y(0)+y_x(0)=0, \quad \tau y(1)+y_x(1)=0,$ σ, τ *constants (including* ∞*),*
Case II. $y(0)=hy(1), \quad hp(0)y_x(0)=p(1)y_x(1),$ $h=$*constant,*
Case III. $y(0)=lp(1)y_x(1),\ lp(0)y_x(0)=-y(1), \quad l=$*constant,*

The number of zeros of the solutions of the equation remains unaltered under this transformation.

The value of the integral $\int_0^1 \frac{\partial \Gamma}{\partial \lambda} u^2 dx$ *is the same as that of the corresponding integral* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$.

COROLLARY I. *In Case I the transformation may be so chosen that the boundary conditions can be written in one of the special forms*

$$(a)\ y(0)=y(1)=0; \quad (b)\ y(0)=y_x(1)=0;$$
$$(c)\ y_x(0)=y(1)=0; \quad (d)\ y_x(0)=y_x(1)=0.$$

§ 6. *Oscillation Theorems for the Sturmian Boundary Conditions.*

We propose now to study oscillation theorems for solutions of the differential equation under the first or sturmian case of the general boundary conditions (26) (27). As was shown in the preceding section this case, characterized by the relations $B_{12}=B_{34}=0$ $(A_{12}=A_{34}=0)$, may be reduced to a study of the equation

$$(py_x)_x+G(x,\lambda)y=0, \tag{41}$$

under the boundary conditions

$$\sigma y(0)+y_x(0)=0, \quad \tau y(1)+y_x(1)=0, \tag{42}$$

where σ and τ are constants, the important special forms

$$\left.\begin{aligned}&(a)\ y(0)=y(1)=0; \quad (b)\ y(0)=y_x(1)=0;\\ &(c)\ y_x(0)=y(1)=0; \quad (d)\ y_x(0)=y_x(1)=0;\end{aligned}\right\} \tag{43}$$

being considered as included for the values 0 and ∞. It was further shown that the boundary conditions may always be reduced to one of these special forms.

We proceed now to a study of the nature of the dependence of λ on σ and τ. If $\bar{y}$ and $\bar{\bar{y}}$ are defined as the following fundamental solutions of (41),

$$\bar{y}(0,\lambda)=0, \bar{y}_x(0,\lambda)=1; \quad \bar{\bar{y}}(0,\lambda)=1, \bar{\bar{y}}_x(0,\lambda)=0, \tag{44}$$

any solution $y(x)$ can be written $y=c_1\bar{y}+c_2$. For determination of c_1 and c_2 we have on substitution in (42),

$$\sigma c_2+c_1=0, \quad \tau[c_1\bar{y}(1,\lambda)+c_2\bar{\bar{y}}(1,\lambda)]+c_1\bar{y}_x(1,\lambda)+c_2\bar{\bar{y}}_x(1,\lambda)=0.$$

A necessary and sufficient condition that there be values of c_1, c_2 satisfying the equations is that

$$\begin{vmatrix} 1 & \sigma \\ \tau\bar{y}(1,\lambda)+\bar{y}_x(1,\lambda) & \tau\bar{\bar{y}}(1,\lambda)+\bar{\bar{y}}_x(1,\lambda) \end{vmatrix}=0.$$

Hence each of the parameters σ, τ is a one-valued function of the other and of λ. If we assume that a zero of the solution moves continuously to the right or left, and note what happens as two consecutive zeros pass through the point $x=0$, we see geometrically that for any given σ the condition $\sigma y(0)+y_x(0)=0$ must be satisfied at some stage of the process. It is possible by analysis to ascertain precisely what happens when σ or τ passes over the range $-\infty$, $+\infty$. Corresponding to $\sigma=-\infty$, 0, $+\infty$ one has respectively $y(0)=0$, $y_x(0)=0$, $y(0)=0$, and we shall show that under proper conditions one and only one zero of $y(x)$ is lost or gained by the process. A similar change in τ has a corresponding result.

If in formula (13) we substitute from (42) and the formulae

$$\sigma\frac{\partial y(0)}{\partial\lambda}+\frac{d\sigma}{d\lambda}y(0)+\frac{\partial y_x(0)}{\partial\lambda}=0, \quad \tau\frac{\partial y(1)}{\partial\lambda}+\frac{d\tau}{d\lambda}y(1)+\frac{\partial y_x(1)}{\partial\lambda}=0,$$

obtained from (42) by differentiation with regard to λ, we obtain the relation

$$\int_0^1\frac{\partial G}{\partial\lambda}y^2dx=-p(0)\frac{d\sigma}{d\lambda}y^2(0)+p(1)\frac{d\tau}{d\lambda}y^2(1). \tag{45}$$

Either of the parameters σ, τ may be held fixed. If τ is fixed $\dfrac{d\tau}{d\lambda}=0$, and in order that λ and σ be one-valued functions of one another over the range $-\infty$, $+\infty$ of σ it is then necessary and sufficient that the integral $\displaystyle\int_0^1\frac{\partial G}{\partial\lambda}y^2dx$ have one sign. From such considerations one may deduce a variety of theorems of which the following are examples.

Theorem XIII. *If λ'_m, λ'_{m+1} denote two successive characteristic numbers of a solution of* (41) *for the boundary conditions* 43(a), *then in order that there be for all σ an intermediate value of λ corresponding to which there is precisely one solution of* (41) *for the boundary conditions $\sigma y(0)+y_x(0)=0$, $y(1)=0$, it is necessary and sufficient that the integral $\displaystyle\int_0^1\frac{\partial G}{\partial\lambda}y^2dx$ have one sign for all solutions $y(x)$ concerned.*

THEOREM XIV. *If $\bar{\lambda}_m$, $\bar{\lambda}_{m+1}$ denote two successive characteristic numbers of a solution of* (41) *for the boundary conditions*

$$\sigma y(0)+y_x(0)=0, \quad y(1)=0, \quad \sigma = constant,$$

then in order that there be for all τ an intermediate value of λ corresponding to which there is precisely one solution of (41) (42), *it is necessary and sufficient that the integral $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ have one sign for all solutions $y(x)$ concerned.*

THEOREM XV. *If corresponding to λ', λ'' there are solutions of* (41) (43*a*) *with n_1 and n_2 zeros respectively, and if $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ has one sign, then in the interval λ', λ'' there is one and only one value of λ for which* (41) (42) *has a solution $y(x)$ with n zeros $(n_1 < n < n_2)$.*

We have seen in § 5 that by a transformation of the dependent variable the equation (41) may be thrown into the form

$$(\bar{p}\bar{y}_x)_x + \bar{G}\bar{y} = 0, \quad \int_{\alpha_1}^{\alpha_2} \frac{\partial \bar{G}}{\partial \lambda} \bar{y}^2 dx = \int_{\alpha_1}^{\alpha_2} \frac{\partial G}{\partial \lambda} y^2 dx, \quad \bar{p}(x) > 0,$$

where the boundary conditions (42) assume one of the special forms

$$\bar{y}(\alpha_1) = \bar{y}(\alpha_2) = 0; \quad \bar{y}(\alpha_1) = \bar{y}_x(\alpha_2) = 0; \quad \bar{y}_x(\alpha_1) = \bar{y}(\alpha_2) = 0; \quad \bar{y}_x(\alpha_1) = \bar{y}_x(\alpha_2) = 0.$$

As an extension of *Theorems* IV and IVA we have then the following:

THEOREM XVI. *If a zero of one of the functions $\sigma y + y_x$, $\tau y + y_x$ is held fixed, then with increasing λ the zeros of the other move closer to the fixed zero or further away according as $\int \frac{\partial G}{\partial \lambda} y^2 dx$ is positive or negative, the integration extending over the interval between the zeros.*

From the preceding theorems we can deduce still further results.

THEOREM XVII. *If $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ has one sign, then between any two consecutive λ's corresponding to solutions of* (41) (43*a*) *or of* (41) (43*d*) *there will be one each corresponding to solutions of* (41) (43*b*) *and* (41) (43*c*), *and between two consecutive λ's for* (41) (43*b*) *or* (41) (43*c*) *there will be one each for* (41) (43*a*) *and* (41) (43*d*).

COROLLARY: *Denoting by λ_a and λ_d two adjacent characteristic values for solutions of the problem* (41) (43*a*), (41) (43*d*) *respectively, and by λ_b, λ_c and λ'_b, λ'_c the next greater and next smaller sets for solutions of the problem* (41) (43*b*), (41) (43*c*) *respectively, then if the number of zeros of the solution*

corresponding to λ_a *is* m, *the number corresponding to* λ_d *is* $m-1$. *When the integral is positive the number of zeros corresponding to* λ_b *and* λ_c *is* m, *and to* λ_b' *and* λ_c' *is* $m-1$, *while if it is negative, the reverse is true.*

If we consider the special form* of (41) where $\frac{\partial G}{\partial \lambda} \geqq 0$ and Hypothesis A of § 2 is satisfied while $G(x, -\infty)<0$ and $=-\infty$ in at least some portion of the interval, we can trace the various values of λ corresponding to successive suites of σ and τ, and state

THEOREM XVIII. *There exists under these hypotheses an infinite set of characteristic numbers* $\lambda_1<\lambda_2<\lambda_3\ldots$, *with limit point at* $+\infty$ *only, for which exist solutions of* (41) (42), *the solution* $y_n(x)$ *corresponding to* $\lambda_n(n=1, 2, 3, \ldots)$ *having* $n-1$ *zeros within the interval.*

The results for the orthogonal case (§ 1, II) of the special equation

$$(py_x)_x+(q+\lambda k)y=0, \tag{46}$$

are contained in the foregoing theorem. In the other cases it is preferable to investigate the zeros through a discussion of those of the equation obtained by the transformation used in proving *Theorem* XVI. We have

$$(\bar{p}\bar{y}_x)_x+(\bar{q}+\lambda\bar{k})\bar{y}=0, \quad \bar{p}(x)>0, \quad \bar{y}(0)\bar{y}_x(0)=\bar{y}(1)\bar{y}_x(1)=0,$$
$$\int_0^1(\bar{p}\bar{y}_x^2-\bar{q}\bar{y}^2)\,dx=\int_0^1\bar{k}\bar{y}^2dx=\int_0^1 ky^2dx.$$

If the new equation is of the polar form we see at once that there are precisely two solutions of (46) (42) *with* n *zeros in the interval. If the equation is of the non-definite form there are two integers* n_1, n_2, *such that for* $n<n_1$ *there are no solutions of* (46) (42), *for* $n>n_1$ *there are at least two, while for* $n>n_2$ *there are precisely two* (*Theorems VII, VII A*).

§ 7. *Oscillation Theorems for Case II of the Boundary Conditions.*

We shall next consider the exceptional case $B_{42}\equiv\alpha_4\beta_2-\alpha_2\beta_4=0$ for which, as has been seen in § 5, the boundary conditions may be written

$$y(0)=hy(1), \quad hp(0)y_x(0)=p(1)y_x(1), \quad h=\text{constant}\neq 0. \tag{47}$$

Defining the particular solutions $\bar{y}(x, \lambda)$, $\bar{\bar{y}}(x, \lambda)$ by (44), and substituting the solution $y=c_1\bar{y}+c_2\bar{\bar{y}}$ in (47), we obtain equations for c_1 and c_2,

$$c_2=h[c_1\bar{y}(1, \lambda)+c_2\bar{\bar{y}}(1, \lambda)], \quad hp(0)c_1=p(1)[c_1\bar{y}_x(1, \lambda)+c_2\bar{\bar{y}}_x(1, \lambda)]. \tag{48}$$

A necessary condition for a solution is the equation

$$D\equiv\begin{vmatrix} h\bar{y}(1, \lambda) & h\bar{\bar{y}}(1, \lambda)-1 \\ p(1)\bar{y}_x(1, \lambda)-hp(0) & p(1)\bar{\bar{y}}_x(1, \lambda) \end{vmatrix}=0. \tag{49}$$

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* This is a slightly more general condition than that imposed by Birkhoff. *Loc. cit.*

Since for the two solutions we have the well-known formula

$$p(x)\,[\bar{\bar{y}}_x(x,\lambda)\bar{y}(x,\lambda)-\bar{y}_x(x,\lambda)\bar{\bar{y}}(x,\lambda)]=\text{constant}=-p(0), \tag{50}$$

this condition reduces to

$$h^2p(0)\bar{\bar{y}}(1,\lambda)-2hp(0)+p(1)\bar{y}_x(1,\lambda)=0,$$

from which we get by again using (50)

$$h=\frac{1}{\bar{\bar{y}}(1,\lambda)}\pm\frac{\sqrt{p(1)}\sqrt{-\bar{\bar{y}}_x(1,\lambda)\bar{y}(1,\lambda)}}{\sqrt{p(0)}\bar{\bar{y}}(1,\lambda)}. \tag{51}$$

So long as $\bar{\bar{y}}_x(1,\lambda)$ and $\bar{y}(1,\lambda)$ have opposite signs there will be two values of h for each value of λ; when they have the same sign there will be none. We may distinguish three critical cases for which the solutions pass from real to complex:

(α) If $\bar{y}(1,\lambda)=0$, $\bar{\bar{y}}_x(1,\lambda)\neq 0$, the equation (49) can by means of (50) be written $[h\bar{\bar{y}}(1,\lambda)-1]^2p(1)\bar{y}_x(1,\lambda)=0$, and since $\bar{y}$ and $\bar{y}_x$ can not vanish together, it follows that $h\bar{\bar{y}}(1,\lambda)-1=0$. Hence every element of the determinant D vanishes except that in the lower right-hand corner; it follows then that $c_2=0$. The solution $y(x)\equiv c_1\bar{y}(1,\lambda)$ vanishes at $x=0$, and since $h\neq 0$, we have from (47) that it vanishes also at $x=1$. The function $y(x)$ is therefore a solution of (41) (43a).

(β) If $\bar{y}(1,\lambda)\neq 0$, $\bar{\bar{y}}_x(1,\lambda)=0$, we have in the same way $c_1=0$ and the solution $y\equiv c_2\bar{\bar{y}}$ also a solution of (41) (43d).

(γ) If $\bar{y}(1,\lambda)=0$, $\bar{\bar{y}}_x(1,\lambda)=0$ it follows immediately by the same reasoning that all elements of D vanish. Hence c_1 and c_2 may take on any values and λ is a double characteristic number.

On multiplying equation (41) by y and integrating under the boundary conditions (47) we obtain the relation

$$\int_0^1 py_x^2dx=\int_0^1 Gy^2dx. \tag{52}$$

By substitution in (13) from formulae (47) and the further formulae,

$$h\frac{\partial y(1)}{\partial\lambda}=\frac{\partial y(0)}{\partial\lambda}-\frac{dh}{d\lambda}y(1),\quad \frac{dh}{d\lambda}p(0)y_x(0)+hp(0)\frac{\partial y_x(0)}{\partial\lambda}=p(1)\frac{\partial y_x(1)}{\partial\lambda}, \tag{53}$$

obtained from (47) by differentiation with regard to λ, we obtain the fundamental formula

$$\frac{dh}{d\lambda}=-\frac{h\int_0^1\frac{\partial G}{\partial\lambda}y^2dx}{2p(0)y(0)y_x(0)}=-\frac{h^2\int_0^1\frac{\partial G}{\partial\lambda}y^2dx}{2p(1)y(0)y_x(1)}=-\frac{\int_0^1\frac{\partial G}{\partial\lambda}y^2dx}{2p(0)y(1)y_x(0)}. \tag{54}$$

Let us denote by λ_b a characteristic number of our problem for $h=0$; this is a solution of the sturmian case (41) (43b). Near it, as we know from *Theorem* XVII and its corollary, there is another characteristic number λ_c corresponding to $h=\infty$ and to (41) (43c), the two solutions for λ_b and λ_c having the same number of zeros. The parameter value λ_b may be equal to, greater than, or less than λ_c. Of the aggregate of characteristic numbers for the two problems (41) (43a), (41) (43d) let us denote by λ_{ad} the greatest of those smaller than λ_b (or λ_c) and by λ'_{ad} the smallest of those greater. Within the interval λ_{ad}, λ'_{ad} it follows from *Theorem* XVII that there is but one zero and one infinity of h; in other words, but one solution for each of the problems (41) (43b), (41) (43c). On the hypothesis that $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ has one sign we are now in a position to prove that *in this interval of* λ *the function* $h(\lambda)$ *defined by* (51) *is monotone on both of its branches* (the one including $h=0$ and the other $h=\infty$).

To prove this let us in the first place note that for $h=0$ or $h=\infty$ we have a sturmian case for which existence theorems have already been established, and then show that as h passes through either of these values, $\frac{dh}{d\lambda}$ does not change sign. We see at once from (47) that $y_x(1,\lambda(0))=0$; hence $y(1,\lambda(0))\neq 0$, and it follows from (47) that $y(0,\lambda(h))$ changes sign with h. A reference to the first part of (54) shows that $\frac{dh}{d\lambda}$ will then retain its sign. In the same way $y(1,\lambda(\infty))=0$, $y_x(1,\lambda(\infty))\neq 0$; $y_x(0,\lambda(h))$ changes sign as h goes through infinity and $\frac{dh}{d\lambda}$ retains its sign. This establishes the result since it follows from (54) that while $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ retains one sign the only possibility of $\frac{dh}{d\lambda}$ changing sign is when h or $y(0)$ or $y_x(0)$ changes sign.

By definition $y(0,\lambda(0))=0$, $y_x(1,\lambda(0))=0$, and without loss of generality it may be assumed that $y_x(0,\lambda(0))>0$. The discussion then divides itself into two parts according to

Hypothesis I. $y(1,\lambda(0))>0$. Hypothesis II. $y(1,\lambda(0))<0$.

Under the first hypothesis the number of zeros of y (including that at $x=0$) is even, while under the other it is odd. Roughly speaking, we shall see that a gain or loss of a zero comes when h goes through 0 or ∞. Concerning the function $\lambda(h)$ there is now sufficient data to sketch the graph. *We shall first discuss the problem for the assumption* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx > 0$ *and Hypothesis I.*

It has been proved above that $\frac{dh}{d\lambda}$ can not change sign on either branch of the function. On the branch containing $h=0$, $\lambda=\lambda_b$ we see from (54) that $\frac{dh}{d\lambda}<0$, since for that particular point $y(1)>0$, $y_x(0)>0$. For $h=\infty$, $\lambda=\lambda_c$, the number of zeros is the same as that for $h=0$ (*Corollary*, *Theorem* XVII) and hence under *Hypothesis* I, $y_x(1)$ and $y(0)$ have opposite signs. From the second part of (54) we see then that $\frac{dh}{d\lambda}>0$ on this other branch. Since at the ends of the interval h is not 0, we know that* in cases (α) and (β) either $y(0)$ or $y_x(0)$ will vanish according as λ_{ad}, λ'_{ad} belong to the problem (41) (43a) or (41) (43d); it follows then from (54) that $\frac{dh}{d\lambda}=\infty$.

Under *Hypothesis* II it is readily shown by the same processes that in an interval which we shall call $\bar{\lambda}_{ad}$, $\bar{\lambda}'_{ad}$ to distinguish it from the other, the branch of the function $\lambda(h)$, which contains $h=0$, is monotone increasing and the branch containing $h=\infty$ is monotone decreasing. As λ increases this form of curve will always alternate with that obtained under *Hypothesis* I. When $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx<0$, the second form of curve occurs under *Hypothesis* I, and the first under *Hypothesis* II.

Since by definition the intervals can not overlap, the curves can not overlap. If the upper bound of one interval is the lower bound of the next, we have a double point for λ [cf. (γ) above], and one branch of one curve unites with one of the other to form a function monotone throughout. This is what takes place, for example, in the case of the solutions of the equation† $y_{xx}+\lambda y=0$.

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*In case (γ), $y(0)$ and $y_x(0)$ may be chosen arbitrarily; cf. next succeeding foot-note.

† For this special equation $\bar{y}\equiv\frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}x$, $\bar{\bar{y}}\equiv\cos\sqrt{\lambda}x$, and equation (49) becomes

$$2h-(h^2+1)\cos\sqrt{\lambda}=0,$$

from which we obtain the formula $h=\sec\sqrt{\lambda}\pm\tan\sqrt{\lambda}$. For every positive λ the function h is double-valued. The interval $\lambda_{ad}, \lambda'_{ad}$ is $(2m)^2\pi^2$, $(2m+1)^2\pi^2$, while at $(2m+1/2)^2\pi^2$, h becomes zero on one branch and infinite on the other. The monotone decreasing branch $h_1\equiv\sec\sqrt{\lambda}-\tan\sqrt{\lambda}$ may be considered as joined at both ends to monotone decreasing branches in the next intervals. These intervals $(2m-1)^2\pi^2$, $(2m)^2\pi^2$ and $(2m+1)^2\pi^2$, $(2m+2)^2\pi^2$ are of the type $\bar{\lambda}_{ad}, \bar{\lambda}'_{ad}$. The function h_1 decreases from $+\infty$ to $-\infty$ in the interval $(2m-1/2)^2\pi^2$, $(2m+3/2)^2\pi^2$. The function $h_2\equiv\sec\sqrt{\lambda}+\tan\sqrt{\lambda}$ increases monotonely from $-\infty$ to $+\infty$ in the interval $(m+1/2)^2\pi^2$, $(m+5/2)^2\pi^2$, the branches of adjacent intervals uniting as in the other case. Each of the curves h_1 and h_2 cuts two of the other set orthogonally, the points of intersection occurring at the end points $m^2\pi^2$ of the interval λ_{ad}, λ'_{ad}.

It is now easy to write down oscillation theorems for this case. We note in the first place that under *Hypothesis* I the number of zeros is increased by unity as h goes through zero, and under *Hypothesis* II it is decreased by unity so that for $h>0$ the number of zeros is always even, and for $h<0$ it is odd. There are then two solutions with an even number of zeros in the one case and two with an odd number in the other.

THEOREM XIX. *If in an interval of* λ *there exist two integers* m_1, m_2 *positive or zero such that there are solutions of*

$$(py_x)_x+G(x,\lambda)y=0 \tag{55}$$

and (43a) *with* m_1 *and* m_2 *zeros respectively within the interval* 0, 1, *then under the hypothesis that* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ *has one sign throughout, there are for* $h>0$ *two solutions of* (55) (47) *when* m *is even, and none when* m *is odd* $(m_1 \leq m \leq m_2)$: *for* $h<0$ *there are two solutions when* m *is odd aud none when* m *is even.*

The special case where $\frac{\partial G}{\partial \lambda}>0$ and G becomes negative for $\lambda=-\infty$, and moreover negatively infinite in at least a part of the interval and G becomes positively infinite for $\lambda=+\infty$ in at least a part of the interval, includes the case discussed by Birkhoff and the detailed theorems derived by him hold also here.*

The orthogonal problem (§ 1, II) of the equation

$$(py_x)_x+(q+\lambda k)y=0 \tag{56}$$

is contained in the special case just discussed. In discussing the polar case we note that it follows from the special case of the formula (52),

$$0<\int_0^1 (py_x^2-qy^2)\,dx=\lambda\int_0^1 ky^2 dx, \tag{57}$$

that $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx \equiv \int_0^1 ky^2 dx$ has the same sign as λ. Hence $\lambda=0$ is not included in the range of values (unless one includes the solution $y\equiv 0$).

THEOREM XX. *In the polar case of equation* (56) *there are precisely two solutions satisfying the boundary conditions* (47) *and oscillating* n *times* $(n=0, 1, 2, \ldots.)$.

In the non-definite case we have from *Theorem* VII that for sufficiently large values of $|\lambda|$ the integral on the left of (57) is positive and hence we can state

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* Case II corresponds to III, p. 269 in Birkhoff's article, *loc. cit.*

THEOREM XXI. *For the non-definite case of equation* (56) *there exist two integers* n_1, n_2 ($n_2 \geqq n_1 \geqq 0$) *such that for* $n < n_1$ *there is no solution of* (56) (47) *with* n *zeros in the interval; for* $n > n_1$ *there are at least two, while for* $n \geqq n_2$ *there are precisely two.*

§ 8. *Oscillation Theorems for Case III of the Boundary Conditions.*

There remains the most general of the three normal forms obtained for the boundary conditions, viz.:

$$y(0) = lp(1)y_x(1), \quad lp(0)y_x(0) = -y(1), \quad l = \text{constant} \neq 0. \tag{58}$$

Since the discussion follows the same lines as that of the preceding section it will be abbreviated. If we define the two particular solutions $\bar{y}(x, \lambda)$, $\bar{\bar{y}}(x, \lambda)$ as in (44), we get in place of formulae (49), (51),

$$\left.\begin{aligned} D' &\equiv \begin{vmatrix} lp(1)\bar{y}_x(1, \lambda) & lp(1)\bar{\bar{y}}_x(1, \lambda) - 1 \\ \bar{y}(1, \lambda) + lp(0) & \bar{\bar{y}}(1, \lambda) \end{vmatrix} = 0, \\ l &= \frac{1}{p(1)\bar{\bar{y}}_x(1, \lambda)} \pm \frac{\sqrt{\bar{y}_x(1, \lambda)\bar{\bar{y}}(1, \lambda)}}{\sqrt{p(0)\,p(1)\bar{\bar{y}}_x(1, \lambda)}}. \end{aligned}\right\} \tag{59}$$

The critical values are when $\bar{y}_x(1, \lambda) = 0$ and $\bar{\bar{y}}(1, \lambda) = 0$ and the cases may be classified as before.

(α) If $\bar{\bar{y}}(1, \lambda) = 0$, $\bar{y}_x(1, \lambda) \neq 0$, equation (59) can be written

$$\bar{y}(1, \lambda)\,[lp(1)\bar{\bar{y}}_x(1, \lambda) - 1]^2 = 0 \quad \text{or} \quad lp(1)\bar{\bar{y}}_x(1, \lambda) = 1.$$

It follows that $c_1 = 0$ and the solution $y(x, \lambda) \equiv c_2\bar{\bar{y}}(x, \lambda)$ is a solution of the equation with the sturmian boundary condition (43c).

(β) If $\bar{\bar{y}}(1, \lambda) \neq 0$, $\bar{y}_x(1, \lambda) = 0$ we have $c_2 = 0$ and $y \equiv c_1\bar{y}$, which is a solution of (41) (43b).

(γ) If $\bar{\bar{y}}(1, \lambda) = 0$, $\bar{y}_x(1, \lambda) = 0$, it follows as before that D' vanishes identically and λ is a double parameter value.

To replace formula (53) and (54) we have

$$\frac{\partial y(0)}{\partial \lambda} = \frac{dl}{d\lambda}p(1)y_x(1) + lp(1)\frac{\partial y_x(1)}{\partial \lambda},$$

$$lp(0)\frac{\partial y_x(0)}{\partial \lambda} + \frac{dl}{d\lambda}p(0)y_x(0) = -\frac{\partial y(1)}{\partial \lambda},$$

$$\frac{dl}{d\lambda} = -\frac{l\int_0^1 \frac{\partial G}{\partial \lambda}y^2 dx}{2p(0)y_x(0)y(0)} = \frac{l^2\int_0^1 \frac{\partial G}{\partial \lambda}y^2 dx}{2y(1)y(0)} = \frac{-\int_0^1 \frac{\partial G}{\partial \lambda}y^2 dx}{2p(0)y_x(0)p(1)y_x(1)}. \tag{60}$$

It is readily shown as in § 7 that $\frac{dl}{d\lambda}$ does not change sign as l goes through the values 0 or ∞. Let us denote by λ_a, λ_d two adjacent characteristic numbers for the cases $l=0$ and $l=\infty$, respectively (in other words for (41) (43a), (41) (43d)), by λ_{bc} the greatest of the aggregate of characteristic numbers for (41) (43b) and (41) (43c) which are smaller than λ_a, and by λ'_{bc} the smallest of the aggregate larger. It is readily shown as in the previous section that within the interval λ_{bc}, λ'_{bc}, l is a monotone function on the branch containing $l=0$ and on the branch containing $h=\infty$. Let us consider first the case $l=0$; then $y(0,\lambda(0))=0$, $y(1,\lambda(0))=0$ and we can assume $y_x(0,\lambda(0))>0$. There will be two cases to distinguish according as we make

Hypothesis I. $y_x(1,\lambda(0))>0$; Hypothesis II. $y_x(1,\lambda(0))<0$.

Under *Hypothesis* I the number of zeros is always odd, and when l goes through zero from negative to positive it may be seen from (58) that two zeros of the solution $y(x)$ are lost. Under *Hypothesis* II the number of zeros is even, and, as l increases through zero, two zeros of $y(x)$ are gained. It follows from the last part of (60) that the integral $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ and $\frac{dl(0)}{d\lambda}$ have opposite signs. This fixes the sign of $\frac{dl}{d\lambda}$ on one branch. To fix the sign of $\frac{dl}{d\lambda}$ on the other branch let us consider $\lambda(\infty)\equiv\lambda_d$. Under *Hypothesis* I we can argue from the corollary to *Theorem* XVII that $y(0,\lambda(\infty))$ and $y(1,\lambda(\infty))$ have the same sign, and from the third part of (60) that when $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx>0$, $\frac{dl(\infty)}{d\lambda}$ is positive. Under *Hypothesis* II we can easily prove that the situation is reversed, $\frac{dl}{d\lambda}$ being positive on the branch through $h=0$, and negative on the other. When $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx<0$ the two types of curves are interchanged. With increasing λ we see then that in both cases curves of much the same form as the two varieties in the preceding section alternate with one another. If adjacent intervals have the same end-point, a branch from the one will unite with a branch of the other.*

From these data various theorems may be deduced, of which the following is typical.

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* For the special equation $y_{xx}+\lambda y=0$ one can set up the formula for $l(\lambda)$ as in foot-note, p. 312. It is found that $l=-\csc\sqrt{\lambda}\pm\cot\sqrt{\lambda}/\sqrt{\lambda}$ and that these monotone functions extend from $-\infty$ to $+\infty$, each of the one set cutting two of the other set in double points $\lambda=(n+1/2)^2\pi^2$ of $l(\lambda)$.

THEOREM XXII. *If in an interval of* λ *there exist two integers* $\mu_1 < \mu_2$, *positive or zero, such that there are solutions of* (55) (43*a*) *with* μ_1 *and* μ_2 *zeros respectively within the interval* 0, 1, *then*

(1) *Under the hypothesis that* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ *be positive, there are, when* l *is positive, two solutions of* (55) (58) *which have* $2m$ *or* $2m-1$ *zeros in the interval* ($\mu_1 < 2m-1 < 2m < \mu_2$) *and there are, when* l *is negative, two solutions which have* $2m$ *or* $2m+1$ *zeros* ($\mu_1 < 2m < 2m+1 < \mu_2$).

(2) *Under the hypothesis that* $\int_0^1 \frac{\partial G}{\partial \lambda} y^2 dx$ *be negative, there are, when* l *is positive, two solutions of* (55) (58) *which have* $2m$ *or* $2m+1$ *zeros in the interval* ($\mu_1 < 2m < 2m+1 < \mu_2$) *and there are, when* l *is negative, two solutions which have* $2m$ *or* $2m-1$ *zeros* ($\mu_1 < 2m-1 < 2m < \mu_2$).

One can make the results of this theorem more specific by giving the conditions necessary to characterize the branch of the function $l(\lambda)$ which is involved. Birkhoff has done this for the special case treated by him (*loc. cit.*, p. 269, I, II), but we shall content ourselves with stating that the same classification may be made in the general case treated here.

Theorems for the special equation (56) analogous to *Theorems* XX, XXI, can be at once written down.

BROWN UNIVERSITY, *January*, 1918.